The Moment Generating Function

Gary Schurman, MBE, CFA

October, 2011

The moment generating function ($M_x(t)$) of the random variable x as a function of the variable t is...

$$M_x(t) = \mathbb{E}\Big[e^{tx}\Big] \tag{1}$$

Note that e^{tx} can be approximated around zero using a Taylor Series Expansion. When x changes from zero to a non-zero value the approximate value of Equation (1) at the new value of x via a Taylor Series Expansion is...

$$\begin{aligned}
M_x(t) &= \mathbb{E}\Bigg[M_0(t) + \frac{\delta M_0(t)}{\delta x}\delta x + \frac{1}{2}\frac{\delta^2 M_0(t)}{\delta x^2}\delta x^2 + \frac{1}{6}\frac{\delta^3 M_0(t)}{\delta x^3}\delta x^3 + ... + \frac{1}{n!}\frac{\delta^n M_0(t)}{\delta x^n}\delta x^n\Bigg] \\
&= \mathbb{E}\Bigg[e^{t0} + te^{t0}(x-0) + \frac{1}{2}t^2 e^{t0}(x-0)^2 + t^3 e^{t0}(x-0)^3 + ... + t^n e^{t0}(x-0)^n\Bigg] \\
&= \mathbb{E}\Bigg[1 + tx + \frac{1}{2}t^2x^2 + \frac{1}{6}t^3x^3 + ... + \frac{1}{n!}t^n x^n\Bigg] \\
&= 1 + t\,\mathbb{E}\Big[x\Big] + \frac{1}{2}t^2\,\mathbb{E}\Big[x^2\Big] + \frac{1}{6}t^3\,\mathbb{E}\Big[x^3\Big] + ... + \frac{1}{n!}t^n\,\mathbb{E}\Big[x^n\Big]
\end{aligned} \tag{2}$$

To calculate the first moment of the distribution (i.e. M1, which is the expected value of x) we take the first derivative of Equation (2) with respect to t and evaluate it at t = 0 because...

$$\begin{aligned}
M1 &= \lim_{t\to 0}\frac{\delta M_x(t)}{\delta t} \\
&= \lim_{t\to 0}\Bigg\{\mathbb{E}\Big[x\Big] + t\,\mathbb{E}\Big[x^2\Big] + \frac{1}{2}t^2\,\mathbb{E}\Big[x^3\Big] + ... + \frac{1}{(n-1)!}t^{n-1}\,\mathbb{E}\Big[x^n\Big]\Bigg\} \\
&= \mathbb{E}\Big[x\Big]
\end{aligned} \tag{3}$$

To calculate the second moment of the distribution (i.e. M2, which is the expected value of the square of x) we take the second derivative of Equation (2) with respect to t and evaluate it at t = 0 because...

$$\begin{aligned}
M2 &= \lim_{t\to 0}\frac{\delta^2 M_x(t)}{\delta t^2} \\
&= \lim_{t\to 0}\Bigg\{\mathbb{E}\Big[x^2\Big] + t\,\mathbb{E}\Big[x^3\Big] + ... + \frac{1}{(n-2)!}t^{n-2}\,\mathbb{E}\Big[x^n\Big]\Bigg\} \\
&= \mathbb{E}\Big[x^2\Big]
\end{aligned} \tag{4}$$

And so on...

The Normal Distribution

The normal distribution is a distribution of continuous random variables. If the random variable x is continuous the moment generating function of the random variable x where $f(x)$ is the probability density function can be defined as...

$$M_x(t) = \int\limits_{-\infty}^{\infty} e^{tx} f(x)\,\delta x \tag{5}$$

The equation for the probability density function of the normal distribution with mean m and variance v is...

$$f(x) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(x-m)^2} \tag{6}$$

If we combine Equations (5) and (6) from above we can write the equation for the moment generating function of the normal distribution as...

$$M_x(t) = \int\limits_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(x-m)^2} \delta x \tag{7}$$

To solve Equation (7) we will first define a new random variable z to be the normalized random variable x. The equation for the random variable z is...

$$z = \frac{x-m}{\sqrt{v}} \tag{8}$$

The random variable x as a function of the new random variable z, which was defined in Equation (8) above, is...

$$x = m + z\sqrt{v} \tag{9}$$

The derivative of Equation (9) with respect to the random variable z is...

$$\frac{\delta x}{\delta z} = \sqrt{v} \tag{10}$$

To solve Equation (7) we will next replace the variable x with the variable z as defined in Equations (8), (9) and (10) above. Equation (7) rewritten to be a function of the variable z is...

$$\begin{aligned} M_x(t) &= \int\limits_{-\infty}^{\infty} e^{t(m+z\sqrt{v})} \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2}z^2} \left| \frac{\delta x}{\delta z} \right| \delta z \\ &= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2}z^2 + z\sqrt{v}\,t} e^{mt} \left| \sqrt{v} \right| \delta z \\ &= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 + z\sqrt{v}\,t} e^{mt} \,\delta z \end{aligned} \tag{11}$$

Noting that after completing the square...

$$-\frac{1}{2}\left(z - \sqrt{v}\,t\right)^2 = -\frac{1}{2}\left(z^2 - 2z\sqrt{v}\,t + vt^2\right) = -\frac{1}{2}z^2 + z\sqrt{v}\,t - \frac{1}{2}vt^2 \tag{12}$$

We can use Equation (12) to rewrite Equation (11), which becomes...

$$\begin{aligned} M_x(t) &= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sqrt{v}\,t)^2} e^{mt} e^{\frac{1}{2}vt^2} \,\delta z \\ &= e^{mt+\frac{1}{2}vt^2} \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sqrt{v}\,t)^2} \,\delta z \\ &= e^{mt+\frac{1}{2}vt^2} \end{aligned} \tag{13}$$

The first derivative of Equation (13) with respect to the variable t is...

$$\frac{\delta M_x(t)}{\delta t} = (m + vt)\, e^{mt+\frac{1}{2}vt^2} \tag{14}$$

The first moment of the distribution will be the limit of Equation (14) as t goes to zero

$$\begin{aligned} \mathbb{E}\Big[x\Big] &= \lim_{t \to 0} \frac{\delta M_x(t)}{\delta t} \\ &= (m + v(0))\, e^{m(0)+\frac{1}{2}v(0)^2} \\ &= m \end{aligned} \tag{15}$$

The second derivative of Equation (13) with respect to the variable t is...

$$\frac{\delta^2 M_x(t)}{\delta t^2} = v\, e^{mt+\frac{1}{2}vt^2} + (m+vt)^2 e^{mt+\frac{1}{2}vt^2} \tag{16}$$

The second moment of the distribution will be the limit of Equation (16) as t goes to zero

$$\begin{aligned}\mathbb{E}\Big[x^2\Big] &= \lim_{t\to 0} \frac{\delta^2 M_x(t)}{\delta t^2} \\ &= v\, e^{m(0)+\frac{1}{2}v(0)^2} + (m+v(0))^2 e^{m(0)+\frac{1}{2}v(0)^2} \\ &= v + m^2 \end{aligned} \tag{17}$$

The Binomial Distribution

The binomial distribution is a distribution of discrete random variables. If the random variable k, which is the number of successes out of n trials, is discrete then the moment generating function of the random variable k where $P(k)$ is the probability mass function can be defined as...

$$M_k(t) = \sum_{k=0}^{n} e^{tk} P(k) \tag{18}$$

The equation for the probability mass function of the binomial distribution with the probability of success equal to p and the probability of failure equal to $1-p$ is...

$$P(k) = \frac{n!}{k!(n-k)!}\, p^k\, (1-p)^{n-k} \tag{19}$$

If we combine Equations (18) and (19) from above we can write the equation for the moment generating function of the binomial distribution as...

$$\begin{aligned} M_k(t) &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, p^k\, (1-p)^{n-k} e^{tk} \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, (pe^t)^k\, (1-p)^{n-k} \end{aligned} \tag{20}$$

Pascal's rule says that...

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, a^k\, b^{n-k} = (a+b)^n \tag{21}$$

If we define $a = pe^t$ and $b = 1-p$ then Equation (20) becomes...

$$M_k(t) = (pe^t + 1 - p)^n \tag{22}$$

The first derivative of Equation (22) with respect to the variable t is...

$$\frac{\delta M_k(t)}{\delta t} = n(pe^t + 1 - p)^{n-1} pe^t \tag{23}$$

The first moment of the distribution will be the limit of Equation (23) as t goes to zero

$$\begin{aligned}\mathbb{E}\Big[k\Big] &= \lim_{t\to 0} \frac{\delta M_k(t)}{\delta t} \\ &= n(pe^0 + 1 - p)^{n-1} pe^0 \\ &= np \end{aligned} \tag{24}$$

The second derivative of Equation (23) with respect to the variable t is...

$$\frac{\delta^2 M_k(t)}{\delta t^2} = n(n-1)(pe^t + 1 - p)^{n-2} pe^t pe^t + n(pe^t + 1 - p)^{n-1} pe^t \tag{25}$$

The second moment of the distribution will be the limit of Equation (23) as t goes to zero

$$\begin{aligned}
\mathbb{E}\left[k^2\right] &= \lim_{t\to 0} \frac{\delta^2 M_k(t)}{\delta t^2} \\
&= n(n-1)(pe^0+1-p)^{n-2}pe^0pe^0 + n(pe^0+1-p)^{n-1}pe^0 \\
&= n(n-1)p^2 + np \\
&= np - np^2 + n^2p^2 \\
&= np(1-p) + n^2p^2
\end{aligned} \tag{26}$$